ANALYTICAL STUDY OF THE HEAT TRANSFER IN A LONGITUDINAL FLOW PAST CYLINDRICAL BODIES OF SMALL RADIUS AT CONSTANT TEMPERATURE

V. R. Borovskii, A. Sh. Dorfman,
V. A. Shelimanov, and O. A. Grechanyi

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We obtain over a wide range of curvature parameters the analytical dependence of the heat transfer in a longitudinal flow past cylindrical bodies of small radius at constant temperature.

In industrial methods for the manufacture of artificial, synthetic, and natural fibers, and also for other materials of cylindrical shape and small radius, considerable importance attaches to the processes of heat transfer and of heat-mass transfer. Studies have shown that the effectiveness of a significant portion of industrial operations depends substantially on the intensity of the processes involving the transport of heat and matter. An analysis of the intensity level of industrial heat and mass transfer processes involved in the production of materials in the form of cylindrical bodies of small radius shows that the industrial heat and mass transfer processes used are of low intensity in comparison with what is theoretically possible.

In a series of experimental studies [1-3] it was shown that when the radius of the cylindrical bodies is decreased (to tenths of a millimeter and less) a sharp growth in the heat transfer coefficients is observed.

A series of analytical studies is also available (see [4-6]) in which heat transfer of cylindrical bodies of small radius is analyzed on the basis of the equations of the boundary layer on a semiinfinite cylinder. Singularities of this problem, associated with taking into account the transverse curvature of the boundary layer, make it impossible to obtain an exact analytical solution. The asymptotic solutions obtained in these papers are valid only in regions of large and small values of a typical curvature parameter, while the approximate solutions, based on applying a one-parameter method for calculating the hydrodynamic and thermal boundary layers, do not have very high accuracy.

We give below an approximate solution of the thermal boundary layer equation for the longitudinal flow past a cylindrical body of small radius. Based on this solution, an expression is obtained for the Nusselt number; an analysis of this expression makes it possible to study the influence of the cylindrical shape of the boundary layer on the heat transfer intensity.

We write the system of equations describing the stationary flow of an incompressible liquid and the transport of thermal energy in the boundary layer on a semiinfinite circular cylinder in the form

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{1}$$

$$\frac{\partial}{\partial x}(ru)+\frac{\partial}{\partial r}(rv)=0, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}=\frac{\nu}{\mathrm{Pr}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3}$$

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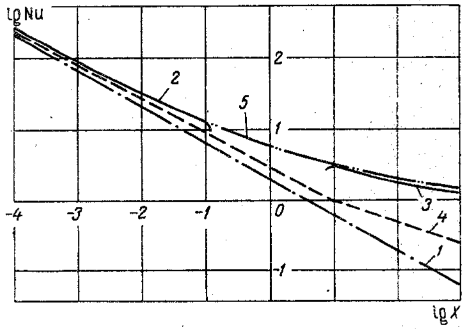

Fig. 1. Dependence of the Nusselt number Nu on the curvature parameter: Curve 1 represents the plate solution; Curve 2 is the Seban, Bond, Kelly solution from [4]; Curve 3 is the Glauert, Lighthill, Bourne solution from [6]; Curve 4 is based on expression (17); Curve 5 is based on expression (24).

with the boundary conditions

$$u = v = 0, \quad T = T_w \text{ for } r = r_0, \qquad u \to U_\infty, \quad T \to T_\infty \text{ for } r \to \infty. \tag{4}$$

We employ next the semi-integral method developed in [7, 8]. In equation (3) we change over to Prandtl–Mises variables:

$$\frac{\partial T}{\partial x} = \frac{\nu}{\Pr}\frac{\partial}{\partial \psi}\left(r^2 u \frac{\partial T}{\partial \psi}\right). \tag{5}$$

Here we have introduced the stream function ψ, satisfying the relations

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial \psi}{\partial x}. \tag{6}$$

As the expression for the longitudinal velocity component u in Eq. (5) we must use, in accordance with the scheme of the semi-integral method, a one-parameter solution of Eqs. (1) and (2), which has the form [5]

$$\frac{u}{U_\infty} = \begin{cases} [\beta(x)]^{-1} \ln \frac{r}{r_0} & \text{for } r < \delta + r_0 \\ 1 & \text{for } r > \delta + r_0, \end{cases} \tag{7}$$

where $\delta = r_0(e^{\beta(x)} - 1)$ is the thickness of the hydrodynamic boundary layer; $\beta(x)$ is a form parameter, determined from the equation

$$\frac{4\nu}{U_\infty r_0^2} = \beta^{-2}\left[(2\beta^2 - 3\beta + 2)e^{2\beta} - \beta - 2\right]\frac{d\beta}{dx}. \tag{8}$$

We now calculate the coefficient of $\partial T/\partial\psi$ on the right side of Eq. (5), taking Eq. (7) into account; we obtain the following expression:

$$r^2 u = \frac{r_0^2 U_\infty}{\beta(x)}\left(\frac{r}{r_0}\right)^2 \ln\frac{r}{r_0}. \tag{9}$$

Substituting the value of u from Eqs. (6) and integrating, we obtain

$$\psi = \frac{r_0 U_\infty}{4\beta(x)}\left[\left(\frac{r}{r_0}\right)^2\left(2\ln\frac{r}{r_0} - 1\right) + 1\right]. \tag{10}$$

To proceed it is necessary to solve Eq. (10) for r/r_0 and to express r^2u as a function of ψ. Since Eq. (10) is transcendental, we solve it approximately.

Consider Eq. (9) for $(r - r_0)/r_0 \ll 1$. Expanding the functions (9) and (10) and taking note of the first nonvanishing terms, we easily find that

$$r^2 u = 2^{\frac{1}{2}}\left[\frac{r_0^2 U_\infty}{\beta(x)}\right]^{\frac{1}{2}}\psi^{\frac{1}{2}}. \tag{11}$$

The expression (11) is found to be sufficiently accurate only in regions in which the parameter $\varkappa = \delta(x)/r_0$ is small. However, in a region far from the beginning of the boundary layer development, wherein $\varkappa > 1$, the accuracy of the expression (11) decreases with an increase in the distance from the cylinder surface.

If we introduce the new dimensionless variables

$$\xi = \sqrt{2}\int_0^X [\beta(x)]^{-\frac{1}{2}}dx, \quad \varphi = \frac{\psi}{r_0^2 U_\infty}, \quad X = \frac{\nu x}{r_0^2 U_\infty}, \tag{12}$$

we can reduce Eq. (5) with the boundary conditions (4) to the form

$$\frac{\partial T}{\partial \xi} = \frac{1}{\Pr}\frac{\partial}{\partial \varphi}\left(\varphi^{\frac{1}{2}}\frac{\partial T}{\partial \varphi}\right), \tag{13}$$

$$T = T_W \text{ for } \varphi = 0; \quad T \to T_\infty \text{ for } \varphi \to \infty; \quad T = T_\infty \text{ for } \xi = 0. \tag{14}$$

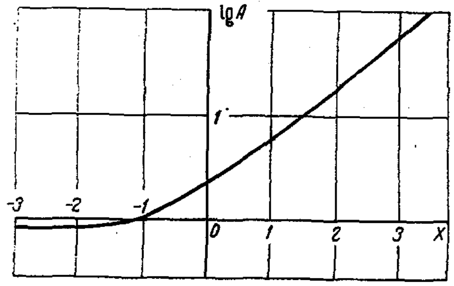

Fig. 2. Dependence of A on X = log $[(\nu x/U_\infty)r_0^2]$.

We now apply a Laplace transformation with respect to ξ to equation (13). Moreover the equation for the dimensionless temperature difference $\Theta = (T-T_\infty)/(T_W-T_\infty)$ is transformed with the aid of the substitution $\varphi_1 = \varphi^{3/4}$ into the following equation of Bessel type:

$$\frac{d^2\Theta_L}{d\varphi_1^2} + \frac{1}{3}\frac{1}{\varphi_1}\frac{d\Theta_L}{d\varphi_1} - \left(\frac{4}{3}\right)^2 \Pr S\Theta_L = 0.$$

Its solution, with the boundary conditions (14) taken into account, has the form

$$\Theta_L(S, \varphi) = \frac{2(S\Pr)^{\frac{1}{6}}\varphi^{\frac{3}{4}}}{S\left(\frac{3}{2}\right)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}.$$

Taking the inverse transform, we obtain the solution of equations (13) and (14) in the form

$$\frac{T-T_\infty}{T_W-T_\infty} = \frac{\Gamma\left(\frac{1}{3}, Z\right)}{\Gamma\left(\frac{1}{3}\right)}, \tag{15}$$

where

$$Z = \frac{\Pr\varphi^{\frac{3}{2}}}{\left(\frac{3}{2}\right)^2\xi}. \tag{16}$$

The resulting Nusselt number, based on the relation (15), has the form

$$\mathrm{Nu} = \frac{-2\pi\lambda\left(r\frac{\partial T}{\partial r}\right)_{r=0}}{\lambda(T_W-T_\infty)} = \frac{\pi}{\Gamma\left(\frac{1}{3}\right)} 3^{\frac{1}{3}}\cdot 2^{\frac{4}{3}}(\Pr)^{\frac{1}{3}}[\beta(X)]^{-\frac{1}{2}}\left[\int\limits^{X}[\beta(x)]^{-\frac{1}{2}}dx\right]^{-\frac{1}{3}}. \tag{17}$$

We note that with the use of Eq. (8) the integral in this expression can be reduced to the form

$$\int\limits_0^X \beta^{-\frac{1}{2}}(x)\,dx = \int\limits_0^{\beta(X)} \beta^{-\frac{1}{2}}\frac{dX}{d\beta}d\beta = \frac{1}{3}\beta^{\frac{1}{2}}\Phi\left(\frac{1}{2}, \frac{3}{2}; 2\beta\right) + \frac{1}{6}\beta^{-\frac{1}{2}}(e^{2\beta}+3) + \frac{1}{3}\beta^{-\frac{3}{2}}(1-e^{2\beta}). \tag{18}$$

Consider the expression (17) in the region $\varkappa \ll 1$. If in this region we use an asymptotic expression for the form parameter (10), namely, $\beta^{-1}(X) \approx 0.278\, X^{-1/2}$, and substitute it into the expression (17), we obtain

$$\mathrm{Nu} \approx 0.668\,\pi(\Pr)^{\frac{1}{3}}X^{-\frac{1}{2}}. \tag{19}$$

This expression coincides with the first term in the asymptotic expansion given in [4] in a region where the curvature parameter is small.

The graph of Nu versus X, calculated from the expression (17), is shown in Fig. 1. For comparison here we give the corresponding curves obtained by asymptotic methods and also the corresponding plate solution. As can be seen from the figure, in the region X < 0.1 the solution obtained here coincides with the asymptotic solution given in [4]. For large values of X the Nu values exceed those for the plate; however in the region $X > 10^2$ the curve does not go beyond the asymptotic solution given by Lighthill, Bourne, and Davies [5, 6]. This is connected with the fact that for large values of the curvature parameter the approximation (11) is less accurate.

Since we are interested in the characteristics of the boundary layer at the surface, we make our results more precise for large values of the curvature parameter by using a weighted form of the r^2u versus ψ relationship given in Eq. (11), i.e., we introduce a weight function A(X) at each boundary layer section; thus

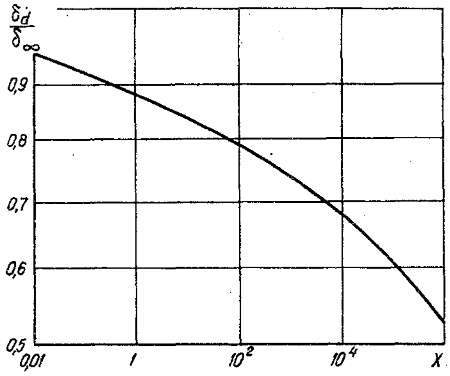

Fig. 3

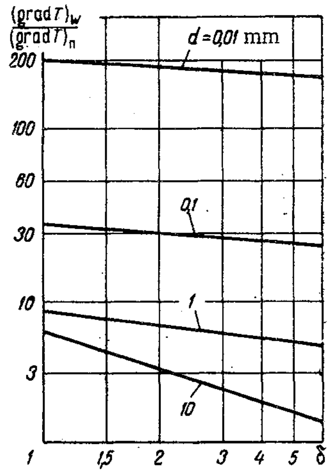

Fig. 4

Fig. 3. Dependence of the relative thickness of the boundary layer on the curvature parameter, $X = (\nu x/U_\infty)r_0^2$.

Fig. 4. Dependence of the ratio of the temperature gradients on cylindrical and flat surfaces on the cylinder diameter and the boundary layer thickness, δ, mm.

$$r^2u = 2^{\frac{1}{2}}\left[\frac{r_0^2 U_\infty}{\beta(x)}\right]^{\frac{1}{2}} A(X)\psi^{\frac{1}{2}}. \tag{20}$$

Replacing r^2u and ψ in this equation by their values from equations (9) and (10), we find that this approximation corresponds to the following approximation of the function appearing in Eq. (9):

$$\left(\frac{r}{r_0}\right)^2 \ln\frac{r}{r_0} = A(X)\left[\left(\frac{r}{r_0}\right)^2\left(2\ln\frac{r}{r_0} - 1\right) + 1\right]^{\frac{1}{2}} \quad \text{for } r - r_0 < \delta(X). \tag{21}$$

In accordance with the semi-integral method the values of $A(X)$ must be determined from some integral condition. In this case it is convenient to use the condition for the equality of the approximate and the exact values of the total heat flow

$$Q = 4\pi^2\rho \int\limits_{r_0}^{\delta(X)} r^2u\frac{\partial T}{\partial r}\,dr,$$

transported by the liquid into each boundary layer section. Equating the values of Q calculated from the approximate and from the exact values of r^2u in accordance with the relation (21), and taking, in a first approximation, the temperature in the boundary layer distributed according to the logarithmic law (7), we obtain the following expression:

$$A(X) = \frac{\int\limits_1^{\delta(X)/r_0} t\ln t\,dt}{\int\limits_1^{\delta(X)/r_0} t^{-1}\left[t^2(2\ln t - 1) + 1\right]^{\frac{1}{2}} dt}. \tag{22}$$

This expression is suitable for $Pr \geq 1$ when the thermal boundary layer is not thicker than the dynamic boundary layer. When $Pr < 1$, the limits of integration in Eq. (22) must be changed to take into account the relationship between the thicknesses of these boundary layers. The dependence of A on X, calculated from this equation, is shown in Fig. 2.

If we now substitute the expression (20) into Eq. (5) and introduce, in place of the first of Eqs. (12), the following transformation for ξ,

$$\xi = \sqrt{2}\int\limits_0^X A(x)\left[\beta(x)\right]^{-\frac{1}{2}} dx, \tag{23}$$

we again arrive at the problem defined by Eqs. (13) and (14), and to a solution of it in the form (15). However, taking into account the new ξ versus X relationship (23), we find that the expression for the Nusselt number now becomes

$$\mathrm{Nu} = \frac{\pi}{\Gamma\left(\frac{1}{3}\right)} 3^{\frac{1}{3}} \cdot 2^{\frac{4}{3}} (\mathrm{Pr})^{\frac{1}{3}} A(X) \beta^{-\frac{1}{2}}(X) \left[\int_0^X \beta^{-\frac{1}{2}}(x) A(x)\, dx\right]^{-\frac{1}{3}}. \quad (24)$$

The results calculated through the use of this function Nu(X), shown in Fig. 1, agree satisfactorily with the exact asymptotic solutions in corresponding limiting regions for the values of the curvature parameter.

It follows from this figure that for identical values of the curvature parameter $(\nu x/U_\infty)r_0^2$ the heat transfer coefficient for a cylindrical body is always greater than for a flat plate. The increase in the heat transfer is reinforced with a decrease in the diameter of the cylindrical body, which, in contrast to the other quantities, enters the parameter as a squared quantity. In particular, for a change in diameter of the cylindrical body from 0.1 mm to 0.01 mm, i.e., a 10-fold decrease, the best transfer coefficient is reinforced by a factor of 5.7, which agrees with the experimental data given in [1-3], whereas the boundary layer thickness is decreased by only 15% (Fig. 3). This means that the growth in the heat transfer intensity is, in the main, due to singularities arising from the cylindrical shape of the boundary layer. The cylindrical shape of the boundary layer causes an increase in the surface of the following layers of the boundary layer. Moreover, the heat flow arriving per unit surface is diminished. This results in a much steeper change in the temperature gradient in the boundary layer on the cylinder in comparison with that on a flat plate. Therefore, as is evident on Fig. 4, with a decrease in the cylinder radius, other conditions being equal, the temperature gradient on the surface increases. It is natural that the influence found to exist on the heat transfer, although not quite substantial, does, in fact, show a decrease in the boundary layer thickness.

It follows from these results that in the manufacture of cylindrically shaped materials of small radius an intensification of heat transfer can be achieved, in the main, not by increasing the flow speed past these cylinders, but by organizing the heat transfer conditions so that the materials would present the smallest possible diameters.

NOTATION

x, r	are the space coordinates;
u, v	are the longitudinal and transverse flow velocity components;
T	is the temperature;
r_0	is the cylinder radius;
T_W	is the surface temperature;
U_∞, T_∞	are the velocity and temperature of undisturbed flow;
ν	is the kinematic viscosity;
Pr	is the Prandtl number;
ψ	is the stream function;
δ	is the boundary layer thickness;
β	is the form parameter;
$\varkappa$	is the curvature parameter;
ξ, φ, X	are the dimensionless variables;
Θ	is the dimensionless temperature;
S	is the parameter of Laplace transformation;
Γ	is the gamma-function;
Z	is the similarity variable;
λ	is the thermal conductivity;
Nu	is the Nusselt number;
Φ	is the degenerated hypergeometric function;
Q	is the heat flux.

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